An Optimal Rate Control Algorithm
for Guaranteed Services in Broadband Networks

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Abstract
Rate control is a known technique usable to prevent congestion caused by multimedia traffics in multiservice networks. Among the proposed control schemes, the connection bit rate can be modified in order to conform a given “leaky bucket”. In this paper we present a new algorithm conforming a leaky bucket, in which the bit rate is varied in order to minimize a linear combination of a twofold objective function respectively related to the difference between the open loop and the current bit rate and to fullness of the token buffer. The obtained solution has some interesting properties by which it is possible to find an approximation that is very close to the optimal solution and very useful for practical purposes because it is given by a linear combination, with constant coefficients, of few samples of the open loop bit rate. First numerical results seem to verify the effectiveness of the proposed method.

1. Introduction
Multimedia services will gain in the next future a wide diffusion, also due to the growth and the enhancement of services provided by the Internet. If a guaranteed quality of service will be provided, traffic generated by these applications can have a severe impact on the network dimensioning and on service robustness. Traffic flows generated by video applications are considered among the most critical ones due to the high peaks and the strong variability of the related bit rate. Different techniques can be applied to reduce such a variability, both when a priori traffic identification is possible (e.g., in case of pre-recorded video sources) and when video signals are transmitted in real time. In particular, smoothing and shaping techniques have been addressed in the literature to this purpose [JR, HR, Lam]. However, one of the problems which limits the applicability of these techniques concerns the possible alteration of the video-quality introduced in shaping and smoothing video sequences.

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Smoothing can be performed in an optimal way, minimizing the playback delay and the receive buffer size for real time sequences as in [LV] or minimizing the number of assigned bandwidth renegotiations and the receive buffer size for stored video as in [JK].

In [HR], traffic shaping occurs directly during the coding phase by varying the quantization parameter in a way that the output rate conforms a given “leaky bucket”. A heuristic relationship is also utilized to change the generated rate according to the fullness of the buffer.

Along this line we propose a procedure to change the rate which is optimal with respect to a suitable cost function evaluating both the video quality of the signal and the efficient utilization of network resources. As to the outline of this paper we first analyze the specific problems arising with the transport of MPEG video sequences on multi-service networks (section 2). In the following section 3, we present the leaky bucket dynamic equations and formulate the optimal control whose solution allows to drive the video coder and to shape the generated traffic. In section 4 a sub-optimal reduced order solution is given which approximates the optimal one with any given precision and therefore an approximate optimal rate control is provided. Finally, in section 5, a numerical application is performed and the results of the proposed sub-optimal procedure are tested against the traditional rate control algorithm showing the effectiveness of the method.

2. Transport of VBR video traffic on broadband networks

Video compression coding techniques are very effective to reduce the total amount of information to be transmitted or to be stored in the video server. The same technique can be also used to modify the video sequences bit rate to simplify properly network access and control operations. As it is known, the MPEG video coding standard [ITU] uses two basic techniques to reduce redundancies of the video signals: block-based motion and transform domain-based compression. To achieve a trade-off between requirements for a random access to the sequences and compression efficiency, three types of frames are defined:

- I-frames, (Intra), are coded without reference to other frames using discrete cosine transform, provide access point to the video sequence and are coded with moderate compression;
- P-frames, (Predictive), are coded with reference to previous I and P frames with compression ratio better than I frames;
- B-frames are coded with reference to the next and the previous I or P frames and having the highest compression ratio.

Due to the use of different kinds of frames and to the intrinsic characteristics of the images, the bit rate of the video sequences is very variable and the peak rate can be much bigger than the average bit rate (e.g. 10 times). Such a variable bit rate of video sequences is obtained, keeping constant the quantization...
parameter of the coder in order to have an almost constant video quality at the output (open loop coding). Sometimes, closed loop coding algorithms [ISO] are instead designed, adjusting the quantization parameter in order to achieve a constant bit rate video sequence; however, in this case the output video quality is variable with time.

As a further aspect, the MPEG sequences are characterised by the long-range dependence property [BST] that makes classical traffic models, based on the use of Markovian not suitable to represent the video traffic. Both rate variability and long range dependence make the transport of video sequences through a broadband network a rather difficult task due to the complexity of the involved access and control operations [RMV].

Moreover, since compressed video traffic is sensitive to packet loss and packet delay, a high Quality of Service (QoS) is required by the network in terms of maximum loss probability and maximum delay and jitter.

In the ATM networks, for example, two network transfer capabilities are mostly involved to carry video traffic with such QoS requirements, i.e., the Deterministic Bit Rate (DBR) and the Statistical Bit Rate (SBR) transfer capabilities. In DBR mode, the peak bit rate is negotiated and reserved for each involved video connection, resulting in a very poor network resource utilisation (e.g., 10%); control and source policing are very simple in this case, as just the peak bit rate has to be allocated and monitored by the network. In the SBR mode, the statistical multiplexing at cell level can improve the resource utilization since it is possible to allocate to each connection a bandwidth lower than the peak bit rate. This bandwidth value, called effective bandwidth [Kel,Lin], is a complex function of the required cell loss probability.

Moreover, at least when the peak rate of the video sequence is in the order of magnitude of the utilized system rate, the relevant effective bandwidth becomes very close to the peak and the resource utilization is still very low. With effective bandwidth we have more complex control and policing operations as both the sustainable rate and the burst duration have to be monitored by the network. Furthermore, as already noted, the above mentioned long range dependence of the video sequence makes difficult a right evaluation of the traffic parameters of the video signals [GW, RMR, RB].

To cope with the previous problems, video sequences can be suitably shaped in order to reduce both burstiness and long range dependence of the coded video sequences.

In particular, signals can be shaped varying the quantization parameter during the coding phase, to reduce bit rate variability. In this case the sequence shape is permanently modified. Obviously, when the quantization parameter is increased, the image can suffer from some quality reduction. For this reason, this variation is operated only in correspondence to the parts of the sequence which are full of actions, so that the human eye may be not able to recognize the quality reduction. This method is proposed in [HR] where, in particular, the quantization parameter is varied in order to make the sequence conform to a leaky bucket, reducing the rate variability of the sequence and making the traffic parameters of the sequence...
more predictable. Since there are many ways to vary the quantization parameter still making the sequence conform to a leaky bucket, a procedure is proposed in section 3 an algorithm to control the quantization parameter, to achieve sequences which are optimal for a given cost function related to the video quality of the signals as well as to the leaky bucket concept.

3. Optimal Rate Control

In MPEG video coding the quantization parameter is inversely related to the emitted bit rate [HR], so we can just focus on the determination of this latter and then from this quantity it is possible to derive the quantization parameter.

In defining an optimization problem concerning the bit rate of a video sequence, it is necessary to consider both the need to provide the video with a given quality and the need to avoid congestion in the network.

Let firstly refer to a constant value \( q \) of the quantization parameter. Let \( R(k) \) be the variable number of bits emitted in the \( k \)-th GoP when the quantization parameter is \( q \) [ \( R(k) \) is called the open loop bit rate]. To avoid network congestion the video bit rate is required to meet a set of constraints called “burstiness constraints”. These constraints can be replaced with the ones which characterize the “leaky bucket” dynamic.

Even meeting such constraints, the video quality has to be preserved, both on the average and on “short periods”. To this aim we formulate an optimization problem where the cost function is a weighted sum of two components. The first one considers the sum of quadratic differences between the open loop bit rate and the unknown optimal shaped rate. The second one considers the state of the credits in the “leaky bucket” and in particular minimizes the sum of quadratic differences between the number of credits and \( b/2 \), where \( b \) is the maximum number of credits. The second term avoids the under-utilization of the allowed bit rate and also the sudden reduction of the shaped rate to its mean value \( r \) in case of long burst of high bit rate, that determines a very negative effect on the short term video quality. In this sense we take into account both average and short term video quality.

3.1 Burstiness constraints

Let us consider the discrete representation of a bit sequence generated by a video coder, where \( R(k) \) is the number of bits emitted in the \( k \)-th GoP, for \( k = 1..N, R(k)\in \mathbb{N} \).

In order to avoid congestion at burst scale, the following set of constraints have to be met [RMV]:

\[
\sum_{i=0}^{h} R(k+i) \leq (h+1)r + b \quad h = 0, 1, \ldots, N-1 \quad k = 1, 2, \ldots, N-h
\]

(3.1)
with \( r \) and \( b \) positive constants which will have a physical interpretation in terms of leaky bucket dynamic. The total number of such constraints is \( N(N+1)/2 \).

The explicit expression of such constraints is:

\[
R(k) \leq r + b \\
R(k)+R(k+1) \leq 2r + b \\
R(k)+R(k+1)+R(k+2) \leq 3r + b \\
\ldots \\
R(1)+R(2)+R(3)+\ldots+R(N) \leq N(r+b)
\]

There are many ways to choose the bit rate flow meeting the constraints (3.1). In [HR] one of these possible ways is proposed. In this paper we suggested to choose the optimal solution with respect to a suitable cost function. The number of feasible solutions is finite and this property guarantees the existence of such an optimal solution. Nevertheless the direct search of the solution itself is not advisable due to the high cardinality of the admissible set. To this regard, denoting by \( G \subseteq \mathbb{N}^N \) the set of the vectors \( (R(1), R(2), \ldots, R(N)) \) meeting constraints (3.1) and by \( \text{card}\{G\} \) the number of these feasible solutions, we have the following result.

**Theorem 3.1.** For the number of the feasible solutions we have

\[
\text{card}\{G\} \geq \binom{r+b+N}{N} \tag{3.2}
\]

Proof. Denoting by \( R = \{ R(1), R(2), \ldots, R(N) \}, R \in \mathbb{N}^N \) and by

\[
F(N,r+b) = \{ R \in \mathbb{N}^N : R(1)+R(2)+\ldots+R(N) \leq r+b \},
\]

we can prove that \( F(N,r+b) \subseteq G \), \( N=1, 2, \ldots, r+b=0, 1, 2, \ldots \).

In fact, if \( R \in F(N,r+b) \) then:

\[
R(k) \leq R(1)+R(2)+\ldots+R(N) \leq r+b \\
R(k)+R(k+1) \leq R(1)+R(2)+\ldots+R(N) \leq r+b \leq 2r+b \\
\ldots \\
R(1)+R(2)+\ldots+R(N) \leq r+b \leq Nr+b
\]

Therefore \( R \in G \) and

\[
\text{card}\{G\} \geq \text{card}\{F(N+1,r+b)\} \tag{3.3}
\]

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Let us now prove the following property:

\[
\text{card}\{F(N, r+b)\} = \sum_{i=0}^{r+b} \text{card}\{F(N-1, i)\} \quad N=1, 2, ..., r+b=0, 1, 2, ... \quad (3.4);
\]

The \((N)\)-th component \(R(N)\) of the vector \(R\) assume \(r+b+1\) different values that is: 0, 1, 2, ..., \(r+b\). For each fixed value of \(R(N)\), the remaining components can assume all possible values such that the sum is equal to \(r+b-R(N)\) producing a number of combinations equal to \(\text{card}\{F(N-1,r+b-R(N))\}\). Summing this number for each possible value of \(R(N)\) the equation (3.4) is proved. From (3.4) we have:

\[
\text{card}\{F(N, r+b)\} = \text{card}\{F(N-1, r+b)\} + \sum_{i=0}^{r+b-1} \text{card}\{F(N-1, i)\} =
\]

\[
\text{card}\{F(N-1, r+b)\} + \text{card}\{F(N, r+b-1)\}
\]

On the other side it is immediate to verify that also the binomial coefficient satisfy a similar iterative equation:

\[
\binom{r+b+N}{N} = \binom{r+b+N-1}{N-1} + \binom{r+b-1+N}{N}
\]

Noting that:

\[
\text{card}\{F(N, 0)\} = 1 = \binom{N}{N} \quad N=1, 2, ...
\]

\[
\text{card}\{F(1, r+b)\} = r+b+1 = \binom{1+r+b}{1} \quad r+b=0, 1, 2, ...
\]

it follows

\[
\text{card}\{F(N, r+b)\} = \binom{r+b+N}{N} \quad N=1, 2, ... \quad r+b=0, 1, 2, ...
\]

Taking (3.3) into account the theorem is proved. \(\square\)

Observe that the lower bound for \(\text{card}\{G\}\) provided by disequation (3.2) quickly increases with respect to both \(N\) and \(r+b\). For instance when \(N=r+b=10\) it results \(\binom{r+b+N}{N} = 184756\) (in practical applications the values of those parameters are in effect much greater).

### 3.2 Leaky bucket dynamic equations

A well-known solution of the traffic control problem is obtained by the introduction of a “leaky bucket” counter, \(\text{LB}(r,b)\), characterised by an incremental rate \(r\) (bits/GoP) and a maximum counter value \(b\) (bits).

In figure 1 we represent this LB solution in a discrete version, where \(R^*\)\((k)\) is the flow of the video sequence offered to the network at the GoP \(k\), \(R(k)\) is the flow emitted by the LB and carried by the network at the same GoP and \(y(k)\) is the number of credits in the buffer at the same GoP.
The dynamical equations of the system are:

\[ y(k+1) = \min\{y(k)+r-R(k), b\} \quad k=1, 2, ..., N \]
\[ R(k) = \min\{y(k)+r, R^\circ(k)\} \quad k=1, 2, ..., N \]

\[ y(k+1) = \min\{y(k)+r-R(k), b\} \quad k=1, 2, ..., N \]
\[ R(k) = \min\{y(k)+r, R^\circ(k)\} \quad k=1, 2, ..., N \]

It is known [HR] that the bit rate flow \( R(k) \) generated by a LB according to equations (3.5) and (3.6) satisfies the burstiness constraints (3.1).

### 3.3 Formulation of the optimal control problem

As suggested in [HR], we assume to drive the MPEG coder in such a way to produce a sequence \( R^\circ(k) \) which conforms to a LB\((r, b)\), which means that the equation (3.6) becomes

\[ R(k) = R^\circ(k) \quad k=1, 2, ..., N. \]

Furthermore we assume that the buffer capacity \( b \) is great enough that equation (3.5) becomes:

\[ y(k+1) = y(k)+r-R(k) \quad k=1, 2, ..., N \]

Setting the initial value of the buffer state equal to the first integer not less than \( b/2 \) (that is about the median value of its capacity):

\[ y(1) = \lfloor b/2 \rfloor \]

we propose to choose the sequence \( R(k) \) and the corresponding sequence \( y(k) \), in order to minimize, on the control interval \([1, N]\), the quadratic deviations respectively referred to the open loop bit rate sequence \( R(k) \) and the median buffer content value \( b/2 \).

In conclusion, we formulate the following optimum control problem:
Problem 3.2. Given the open loop sequence $\overline{R}(k) \in \mathbb{N} \ k=1,\ldots,N$, find the sequence $R(k) \in \mathbb{N} \ k=1, \ldots, N$ and the corresponding sequence $y(k) \in \mathbb{N} \ k=1, \ldots, N$, which satisfy the dynamical equation (3.7), the initial condition (3.8) and which minimize the cost function:

$$J(y, R) = \sum_{k=1}^{N} \left\{ \alpha[R(k) - \overline{R}(k)]^2 + \left[ y(k+1) - b/2 \right]^2 \right\}$$

where $y$ and $R$ denotes the $N$ component vectors:

$$R = (R(1), \ldots, R(N))^T$$
$$y = (y(2), \ldots, y(N+1))^T$$

and $\alpha>0$ is a fixed weight factor.

3.4 Solution of the optimal control problem

For the solution of the Problem 3.2 it is convenient to consider a relaxed problem equivalent to the Problem 3.2 formulated for $y, R \in \mathbb{R}^N$ instead of $y, R \in \mathbb{N}^N$.

Problem 3.3. See Problem 3.2 for $y, R \in \mathbb{R}^N$

In order to solve the Problem 3.3 of the previous subsection let us introduce the following vectors:

$$u = (u(1), \ldots, u(N))^T$$
$$u_r = (u_r(1), \ldots, u_r(N))^T$$
$$x = (x(1), \ldots, x(N))^T$$

where:

$$u(k) = r - R(k)$$
$$u_r(k) = r - \overline{R}(k)$$
$$k = 1, \ldots, N$$

$$x(k) = y(k+1) - y(1) = y(k+1) - b/2$$
$$k = 1, \ldots, N$$

and the lower $N \times N$ triangular matrix

$$M = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}$$

Now we have the following result:

Theorem 3.4. The optimal solution $R^*$ of Problem 3.3 is given by:

$$R^* = r - P(\alpha, N) u_r$$

where

$$r = (r, \ldots, r)^T$$

$$P(\alpha, N) = \alpha (\alpha I + M^T M)^{-1}$$

Obviously for $y^*$ we have:
where
\[ b = (b, \ldots, b)^T \]

Proof. The cost function (3.9) may be written in the form:
\[ J(u, x) = \alpha (u_r - u)^T(u_r - u) + x^T_i x \]

On the other side the dynamical constraints (3.7) are equivalent to the
\[ x = M u \]

Substituting this latter equation into the cost function, the optimal problem reduces to the unconstrained minimisation of the function:
\[ J(u) = u^T (\alpha I + M^T M) u - 2\alpha u_i^T u + \alpha u_r^T u \]

Since \((\alpha I + M^T M)\) is positive definite, the global minimum of the cost function (3.14) is the unique solution of the equation:
\[ \frac{dJ}{du}_{u^*} = [2 u^* (\alpha I + M^T M) - 2 u_i^T] = 0^T \]

that is:
\[ u^* = \alpha (\alpha I + M^T M)^{-1} u_i = P(\alpha, N) u_i \]  \hspace{1cm} (3.15)

From (3.10) and (3.7), (3.8) the results (3.11), (3.13) immediately follow. \(\square\)

**Remark 3.5.** Note that the solution of the Problem 3.2 may be easily obtained from the one of the Problem 3.3 which is given by Theorem 3.4. In fact, taking into account the geometrical properties of the cost function (3.14), it is easy to verify that the optimal solution of the Problem 3.2, in terms of vector \(u \in \mathbb{N}^i\) (where \(i\) is the set of the integer numbers), is located in one of \(2^N\) vertices of the hypercube of \(\mathbb{N}^N\) containing the vector \(u^*\) given by (3.15). Therefore in the following, we will develop our work, only with reference to the relaxed Problem 3.3 and the related solution.

The above Theorem 3.4 gives the optimal solution in a closed form, for fixed \(N\) and \(\alpha\), and as a function of the known quantities \(R(i)\) \(i=1,\ldots,N\).

From (3.11) easily one obtains the optimal bit rate:
\[ R^*(k) = R \left(1 - \sum_{i=1}^{N} p(k,i)\right) + \sum_{i=1}^{N} p(k,i)R(i) \quad k = 1, 2, \ldots, N \]  \hspace{1cm} (3.16)
where \( p(i,j) \) is the generic element of the matrix \( P(\alpha, N) \). In other words, the optimal bit rate is a weighted sum of the \( N+1 \) quantities \( r \) and \( R(i) \), whose coefficients are in general functions of \( N \). By the analysis reported in the next section, we will prove that, for increasing \( N \), few coefficients are significantly different from zero and their values are virtually independent of \( N \).

4. Sub-optimal asymptotic solution
In this section we will study the behavior of the optimal solution when \( N \) increases, in order to analyze some interesting features and in particular an approximate form which allows remarkable simplifications.

Firstly it is useful to yield an expression for \( P(\alpha, N) \) as a linear combination of powers of the constant tridiagonal \( N \times N \) matrix

\[
K = (M^T M)^{-1} = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & .. & .. & .. & -1 \\
0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

Result 4.1. The following equality holds:

\[
P(\alpha, N) = \sum_{k=1}^{N} \frac{\Delta_k(\alpha, N)}{\Delta(\alpha, N)} K^k
\]

where:

\[
\Delta(\alpha, N) = \det [\alpha I + M^T M] = \sum_{k=0}^{N} c_k(N)\alpha^k
\]

\[
\Delta_k(\alpha, N) = (-1)^{k-1} \sum_{m=k}^{N} c_{m-k}(N)\alpha^m \quad k = 1, 2, ..., N
\]

and \( c_k(N), k=0, 1, ..., N \) are suitable positive integer numbers.

Proof. Keeping in mind equations (3.12) (4.2), let us denote by \( H(\alpha, N) \) the \( N \times N \) matrix

\[
H(\alpha, N) = (1/\alpha) P(\alpha, N) \Delta(\alpha, N) = (\alpha I + M^T M)^{-1} \sum_{k=0}^{N} c_k(N)\alpha^k
\]

Noting that each element of \( H(\alpha, N) \) is a polynomial in \( \alpha \) of degree not greater than \( N-1 \), the Taylor series expansion has the form:

\[
H(\alpha, N) = \sum_{i=0}^{N-1} \frac{d^i H(\alpha, N)}{d\alpha^i} \bigg|_{\alpha=0} \frac{\alpha^i}{(i)!}
\]
On the other hand, by direct calculation, it is possible to verify that:

\[
\frac{d H(\alpha,N)}{d\alpha} = i! \sum_{i=0}^{N-1} (-1)^{N-i} c_i(N) K^{i+1} \quad i=1, 2, ..., N-1
\]  

(4.6)

Observing that \( H(0,N) = c_0(N) K \), by substituting (4.6) and (4.5) into (4.4), we deduce:

\[
P(\alpha,N) = \frac{\alpha}{\Delta(\alpha,N)} \sum_{i=0}^{N-1} \frac{\alpha^i}{i!} \sum_{i=0}^{N} (-1)^{i} c_i(N) K^{i+1}
\]

Finally, by a suitable change of summation indices and taking (4.3) into account, equation (4.1) is verified. As far as the value of coefficients \( c_i(N) \), \( k = 0, 1, ..., N \) we will deduce in the Appendix some iterative relations which allow also to verify that these coefficients are positive integers.

**Remark 4.2.** Note that the matrix \( K \), as already observed, is tridiagonal, \( K^2 \) is a matrix with zero elements out of the five principal diagonals and in general \( K^n \) is a matrix with zero elements out of the \( 2n+1 \) principal diagonals; finally \( K^{N-1} \) has not zero elements. This amount to say that, denoting by \( k_n(i,j) \) the \((i,j)\)-th element of the matrix \( K^n \), it results:

\[
k_n(i,j) = 0 \quad |i-j| \geq n+1 \quad n = 1, 2, ..., N-2
\]

Furthermore, the following equalities can be easily verified by direct computation:

\[
k_n(i, i+j) = k_n(i+1, i+1+j) = k_n(i+1, i+1-j) \quad i=n+1, n+2, ..., N-n-1 \quad j = 1, 2, ..., \left\lfloor \frac{N}{2} \right\rfloor - 1
\]

where \( \left\lfloor N/2 \right\rfloor \) denotes the first integer not less than \( N/2 \).

Let us now introduce a sub-optimal solution which is characterized by a more simple representation with respect to the optimal one (3.16), because it contains weighted sums of only \( M \) terms \( R(i) \), with \( M < N \).

Let us consider the \( N\times N \) matrix

\[
P_M(\alpha,N) = \sum_{k=1}^{M} \frac{\Delta_k(\alpha,N)}{\Delta(\alpha,N)} K^k
\]

(4.7)

with \( M < N \).

Using the recursive formulae reported in Appendix, it has been possible to verify numerically that, given \( N \) sufficiently large and \( \varepsilon > 0 \), a minimum number \( M_\varepsilon(\alpha) < N \) exists such that:

\[
\max_{i,j=1,2,...,N} |p(i,j) - p_M(i,j)| \leq \varepsilon
\]

(4.8)
where \( p(i, j) \) and \( p_M(i, j) \) are respectively the \((i, j)\)-th elements of matrices \( P(\alpha, N) \) and \( P_M(\alpha, N) \).

For example, for \( N=40 \) and \( \varepsilon=2.5 \times 10^{-3} \) the \( M_\varepsilon(\alpha) \) values are reported in figure 2 as a function of \( \alpha \).

Therefore it is possible to use \( P_M(\alpha,N) \) instead of \( P(\alpha,N) \) with a negligible error and with a strong reduction of complexity. To this purpose let us note the following points.

**Remark 4.3.** On the basis of Remark 4.2, \( P_M(\alpha,N) \) is characterized by few diagonals different from zero, while \( P(\alpha,N) \) has no zero elements. More exactly it results:

\[
p_M(i, j) = 0 \quad |i-j| \geq M+1 \quad (4.9)
\]

Moreover the same properties of symmetry stressed in Remark 4.2 for the matrices \( K^n \), still hold for the matrix \( P_M(\alpha,N) \), provided \( M<N/2 \):

\[
p_M(i, i+j) = p_M(i, i+j) = p_M(i+1, i+j+1) = p_M(i+1, i-j+1) \quad i = M+1, ..., N-M-1 \quad j = 1, 2, ..., M
\]

Let us now discuss the characteristics of the sub-optimal solution in terms of the bit rate sequence:

\[
R_M = r - P_M(\alpha,N) u_r
\]

**Result 4.4** As far as the approximation for the sub-optimal bit rate sequence, it results:

\[
|R^*(i) - R_M(i)| \leq \varepsilon C \quad i = 1, 2, ..., N
\]
where:

\[ C = \max_{j=1,2,...,N} \left| r - \mathcal{R}(j) \right| \]  

(4.12)

Proof. From (3.11) and (4.11) we have:

\[ R^* - R_M = (P_M(\alpha, N) - P(\alpha, N))u, \]

Considering a suitable norm, from (4.8) and (4.12) it results:

\[ |R^*(i) - R_M(i)| \leq \max_{i=1,2,...,N} |P(i, j) - p(i, j)| \leq \max_{j=1,2,...,N} |r - \mathcal{R}(j)| \leq \varepsilon C \]

By exploiting the symmetry properties of Remark 4.3 we can analyze the structure of the sub-optimal solution.

From (4.11), similarly to (3.16), we deduce:

\[ R_M(k) = r \left( 1 - \sum_{i=1}^{N} p_M(k, i) \right) + \sum_{i=1}^{N} p_M(k, i) \mathcal{R}(i) \quad k=1, 2, ..., N \]

Taking (4.9) into account we have:

\[ R_M(k) = r \left( 1 - \sum_{i=k-M}^{k+M} p_M(k, i) \right) + \sum_{i=k-M}^{k+M} p_M(k, i) \mathcal{R}(i) \quad k=M+1, ..., N-M \]

Furthermore, considering (4.10), we obtain:

\[ R_M(k) = r(1 - q(\alpha, N)) + l_M(0) \mathcal{R}(k) + \sum_{i=1}^{M} l_M(i) [\mathcal{R}(k-i) + \mathcal{R}(k+i)] \quad k=M+1, ..., N-M \]  

(4.13)

where

\[ l_M(i) = p_M(k, k+i) \quad k=M+1, ..., N-M \]

\[ i=0, 1, ..., M \]

\[ q(\alpha, N) = l_M(0) + 2 \sum_{i=1}^{M} l_M(i) \]

In other words, for \( N \) and \( \alpha \) fixed, the \( M+1 \) coefficients \( l_M(i) \) do not depend on \( k \).

As a conclusion, for a given value of \( \alpha \), it is possible to find good approximation of the optimal solution simply considering a linear combination with constant coefficients, of few samples of the open loop bit rate.

In figure 3 the proposed method is represented. The block B represents a buffer able to make available a sequence of \( 2M+1 \) samples of the open loop bit rate \( \mathcal{R} \); the block L performs the linear operation which generates \( R_M \) according to (4.13).
5. Numerical examples

We applied the proposed procedure to two 1000 frames long video sequences, which correspond to about $N=83$ GoP, encoded at University of Wuerzburg using the UC Berkeley MPEG-1 software encoder [Gon]. In figure 4 we report three curves for each sequence. The black curve refers to the original open loop bit rate $R$, characterized by an average bit rate equal to $m$ and a peak bit rate equal to $p$. The light gray curve represents the output $R$ of a Leaky Bucket with parameters $r$ and $b$ when the input is the open loop bit rate $R$ and it is generated by the equations:

$$y(k+1) = \min \{y(k) + r - R(k), b \}$$  
$$R(k) = \min \{y(k) + r, R(k)\}$$

The mean gray curve represents the sub-optimal shaped bit rate for $a=1$ and $e=2.5 \times 10^{-3}$, generated according to (4.13); note that, as it results from figure 2, for this case it turns out to be $M=5$. The expression of $R_5$ which particularizes (4.13) is the following:

$$R_5(k) = 0.9r + 0.5528 R(k) - 0.1708 [R(k-1) + R(k+1)] - 0.0652 [R(k-2) + R(k+2)] - 0.0249 [R(k-3) + R(k+3)] - 0.0095[R(k-4) + R(k+4)] - 0.0036[R(k-5) + R(k+5)]$$

As it is shown in figure 4, the bit rate variability and the duration of the long burst are reduced both by both the Leaky Bucket and the sub-optimal shaping algorithm, but in different ways. The sub-optimal shaping avoids the sharp reduction to $r$ that characterizes the Leaky Bucket, maintaining a reduced bit rate variability. On the other hand, when the open loop bit rate is lower than $r$, the Leaky Bucket follows exactly $R$, while the bit rate is incremented by the sub-optimal shaping.

Further tests should be performed in order to evaluate the proposed method from objective and subjective quality point of views and to compare it with other possible shaping algorithms.
6. Conclusions
In the paper a method for shaping VBR MPEG video sequences has been presented. The shaped video bit rate is obtained as solution of an optimization problem where the objective is to maintain, as much as possible, a good video quality both over short and long term, subject to the burstiness constraints, i.e., leaky bucket equations to avoid network congestion. The optimal solution has some interesting properties that allow to define an approximate solution. This latter is very close to the optimal solution and is very interesting for practical purposes because it is given by a linear combination, with constant coefficients, of few samples of the open loop bit rate. First numerical results seems to confirm the effectiveness and the interest of the proposed method.

![Figure 4. Bit rate and traffic characteristics for open loop, optimal shaping and LB output](image)

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An Optimal Rate Control Algorithm for Video Connections in Broadband Networks

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Appendix

To calculate $D(a, N)$ and $D_N(a, N)$ for different $N$ and $a$ values, according to (4.2) and (4.3) it is necessary to know the expressions of coefficients $c_i(N)$. To this purpose, simple recursive relations have been found that allow to compute $c_i(N)$ for increasing $N$. We observe that:

$$\Delta(a, N) = \det [\alpha I + M^T M] = \det [\alpha M^{-1} + M^T] \det [M] = \det [\alpha M^{-1} + M^T]$$

Recalling that $M$ is a lower triangular matrix, it can be easily deduced that:

$$M^{-1} = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
-\alpha & 1 & \ldots & 0 \\
0 & -\alpha & 1 & 0 \\
0 & 0 & -\alpha & 1
\end{bmatrix}$$

Therefore we have:

$$\Delta(a, N) = \det \begin{bmatrix}
1 + \alpha & 1 & \ldots & 1 \\
-\alpha & 1 + \alpha & 1 & 1 \\
0 & -\alpha & 1 + \alpha & 1 \\
0 & 0 & -\alpha & 1 + \alpha
\end{bmatrix}$$

$$= c_0(N) + \alpha c_1(N) + \alpha^2 c_2(N) + \alpha^3 c_3(N) + \ldots + \alpha^{N-1} c_N(N)$$

Let us define:

$$\Gamma(a, N) = \det \begin{bmatrix}
1 & 1 & \ldots & 1 \\
-\alpha & 1 + \alpha & 1 & 1 \\
0 & -\alpha & 1 + \alpha & 1 \\
0 & 0 & -\alpha & 1 + \alpha
\end{bmatrix} = $$

$$= d_0(N) + \alpha d_1(N) + \alpha^2 d_2(N) + \alpha^3 d_3(N) + \ldots + \alpha^{N-1} d_{N-1}(N)$$

Obviously we have:

$$\Delta(a, N) = (1 + \alpha) \Delta(a, N - 1) + \alpha \Gamma(a, N - 1)$$

$$\Gamma(a, N) = \Delta(a, N - 1) + \alpha \Gamma(a, N - 1)$$

from which it is possible to deduce the following iterative equations:

$$c_i(N) = c_i(N - 1) + c_{i-1}(N - 1) + d_{i-1}(N - 1) \quad d_i(N) = c_i(N - 1) + d_{i-1}(N - 1) \quad i = 1, 2, \ldots, N - 1 \quad N = 2, 3, \ldots$$

The above equations can be used from the initial conditions

$c_0(N) = 1, N = 1, 2, 3, \ldots \quad c_1(1) = 1 \quad d_0(N) = 1, N = 1, 2, 3, \ldots \quad d_1(1) = 0$

Furthermore by direct calculation it results

$c_N(N) = 1 \quad N = 1, 2, 3, \ldots \quad \Gamma$
References


